

Two-sided reflected Markov-modulated Brownian motion with applications to fluid queues and dividend payouts

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Abstract

In this paper we study a reflected Markov-modulated Brownian motion with a two sided reflection in which the drift, diffusion coefficient and the two boundaries are (jointly) modulated by a finite state space irreducible continuous time Markov chain. The goal is to compute the stationary distribution of this Markov process, which in addition to the complication of having a stochastic boundary can also include jumps at state change epochs of the underlying Markov chain because of the boundary changes. We give the general theory and then specialize to the case where the underlying Markov chain has two states. Moreover, motivated by an application of optimal dividend strategies, we consider the case where the lower barrier is zero and the upper barrier is subject to control. In this case we generalized earlier results from the case of a reflected Brownian motion to the Markov modulated case.

Keywords: Markov modulation, Brownian motion, dividend payout, two sided reflection

1. Introduction

A double sided reflected process, say at zero from below and some positive level b , is a reasonable model for a storage process where the stored quantity has to be nonnegative and the buffer size is limited. When borrowing or backlogging is allowed, then the lower barrier could also be negative. There is a huge literature on such processes, in particular

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when the driving process (before reflection) is Brownian motion. Less attention is given to the case where the boundaries are themselves stochastic processes. For most papers on this topic the focus was on showing the existence and uniqueness of solution of the related Skorohod problem. A recent study which refers to many of the earlier results in this particular direction is [10] where the focus is on multidimensional models. For the one dimensional double sided reflection (non-modulated case), we mention the important results reported in [9] and references therein.

Very little work is done related to the computation of the stationary distribution of such processes when more explicit stochastic structure is assumed, especially when the boundaries are not smooth. One example of such a study is given in [8] where the driving process is Lévy and there is only one lower boundary which increases linearly and then drops back to zero at arrival epochs of a Poisson process.

We are not aware of any results for the case where the boundary together with the driving process are jointly modulated by some other process. This is motivated by situations in which the buffer size and the allowed backlog are allowed to change from time to time as a response to changes in the driving process which are caused by changes in an underlying environment.

In this paper we model the environment as a finite state space irreducible continuous time Markov chain. When in a given state, our process behaves like a two sided reflected Brownian motion with drift and diffusion coefficient as well as lower and upper boundaries which are allowed to depend on this state. The main goal is to give a computational scheme for computing the joint stationary distribution of the buffer content and the state of the underlying environment.

The paper is organized as follows. In Section 2 we present the general model and provide some preliminary results. Section 3 is about the stationary joint distribution of the buffer content and the underlying environment. Section 4 specialized the results to various cases where the lower barrier is zero (no backlog) and the underlying environment changes between two states. Under some conditions, for this case we also show how to compute the distribution of some regenerative epoch associated with this process. Finally, in Section 5 we generalize results of [6], who considered the upper barrier as a cutoff point above which a company must pay dividends that are modeled by the regulating process at this upper barrier.

2. Model

Let $W = \{W(t) | t \geq 0\}$ and $J = \{J(t) | t \geq 0\}$ be two independent processes where W is a Wiener process (a standard Brownian motion) and J is an irreducible and homogeneous continuous time Markov chain with state space $E = \{1, \dots, N\}$. We assume that J has right continuous sample paths and we denote by $\mathbf{Q} = (q_{ij})$ its rate transition matrix, by $\vec{\pi} = (\pi_i)$ its stationary distribution and define $\mathbf{P} = \text{diag}[\vec{\pi}]$. For each $i \in E$ we let $a(i) \leq b(i)$ be two finite real numbers which define the upper and lower barriers when in state i .

It is standard to show that there is a unique process (Z, L, U) satisfying

$$Z(t) = Z(0) + \int_0^t \sigma(J(s)) dW(s) + \int_0^t \mu(J(s)) ds + L(t) - U(t) \quad (1)$$

where $a(J(t)) \leq Z(t) \leq b(J(t))$ for each $t \geq 0$, L and U are nondecreasing right continuous processes with $L(0-) = U(0-) = 0$,

$$\int_0^\infty (Z(s) - a(J(s))) dL(s) = 0 \quad \text{and} \quad \int_0^\infty (b(J(s)) - Z(s)) dU(s) = 0. \quad (2)$$

Z is the two-sided (Skorohod) reflection of the modulated process

$$X(t) = Z(0) + \int_0^t \sigma(J(s)) dW(s) + \int_0^t \mu(J(s)) ds \quad (3)$$

at the modulated barriers $[a(i), b(i)]$, $i \in E$. We denote by $\kappa = \sum_i \mu(i) \pi_i$ the asymptotic drift of the process $X(t)$.

Although Z is not Markovian, (Z, J) is. Let us identify its generator.

Let $f(w, i)$ be a bounded twice continuously differentiable function in w satisfying $f'(a(i), i) = f'(b(i), i) = 0$. We note that clearly there always is a twice continuously differentiable $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $h(w, i) = f(w, i)$. The generalized Itô formula for semimartingales (e.g. Theorem 33 on p. 8 of [11]) now implies after some obvious manipulations that

$$\begin{aligned} f(Z(t), J(t)) &= f(Z(0), J(0)) + \int_0^t \sigma(J(s)) f'(Z(s), J(s)) dW(s) \\ &+ \int_0^t \frac{\sigma^2(J(s))}{2} f''(Z(s), J(s)) + \mu(J(s)) f'(Z(s), J(s)) ds \\ &+ \int_0^t f'(Z(s), J(s)) d(L^c(s) - U^c(s)) + \sum_{0 < s \leq t} \Delta f(Z(s), J(s)) \end{aligned} \quad (4)$$

where for càdlàg functions of bounded variation on compact sets g we denote $g(s-) = \lim_{u \uparrow s} g(u)$, $\Delta g(s) = g(s) - g(s-)$ and $g^c(s) = g(s) - \sum_{0 < u \leq s} \Delta g(u)$. Now, observe that $f'(a(i), i) = 0$ and the first relation in (2) imply that

$$\int_0^t f'(Z(s), J(s)) dL^c(s) = \int_0^t f'(a(J(s)), J(s)) dL^c(s) = 0 \quad (5)$$

and similarly $\int_0^t f'(Z(s), J(s)) dU^c(s) = 0$.

Next, note that the only way that s can be a jump epoch of $f(Z(\cdot), J(\cdot))$ is if it is a jump epoch of J . If $a(J(s)) \leq Z(s-) \leq b(J(s))$ then clearly $Z(s) = Z(s-)$. If either $Z(s-) < a(J(s))$ or $Z(s-) > b(J(s))$, then in the first case necessarily $Z(s) = a(J(s))$ and in the second $Z(s) = b(J(s))$, therefore we can always write $Z(s) = a(J(s)) \vee Z(s-) \wedge b(J(s))$. This implies that

$$\Delta f(Z(s), J(s)) = \hat{f}(Z(s-), J(s)) - \hat{f}(Z(s-), J(s-)), \quad (6)$$

where we used the notation $\hat{f}(w, i) = f(a(i) \vee w \wedge b(i), i)$. Now for each $i \neq j$ let N_{ij} be independent Poisson process with rate q_{ij} , independent of W , such that if an arrival finds J in state i then it instructs J to jump to j and otherwise nothing happens. Now,

$N_{ij}(s) - q_{ij}s$ is a martingale with respect to the filtration generated by W and N_{ij} for $i \neq j$ and $\hat{f}(Z(s), i)$ are bounded processes. Hence,

$$\int_{[0,t]} (\hat{f}(Z(s-), j) - \hat{f}(Z(s-), i)) 1_{\{J(s-)=i\}} d(N_{ij}(s) - q_{ij}s) \quad (7)$$

is a martingale and thus, by summing over $i, j \in E$, with $i \neq j$, and noting that $q_{ii} = -\sum_{j \neq i} q_{ij}$, this implies that

$$\sum_{0 \leq s \leq t} \Delta f(Z(s), J(s)) - \int_0^t \sum_{j \in E} q_{J(s), j} \hat{f}(Z(s-), j) ds \quad (8)$$

is a martingale with respect to the filtration generated by W and the Poisson processes N_{ij} , but also with respect to the filtration generated by W and J , since it is adapted to that filtration and N_{ij} are non-anticipative with respect to it (see also [12], Lemma V.21.13). Since $f'(w, i)$ are locally bounded, it follows that the following is a martingale with respect to the filtration generated by W and J :

$$\begin{aligned} M(t) &= \sum_{0 \leq s \leq t} \Delta f(Z(s), J(s)) - \int_0^t \sum_{j \in E} q_{J(s), j} \hat{f}(Z(s), j) ds \\ &\quad + \int_0^t \sigma(J(s)) f'(Z(s), J(s)) dW(s) . \end{aligned}$$

Rewriting equation (4) in terms of the above martingale we have

$$f(Z(t), J(t)) - f(Z(0), J(0)) = \int_0^t \mathcal{A}f(Z(s), J(s)) ds + M(t)$$

where we denoted by $\mathcal{A}f(z, i)$ the following operator

$$\mathcal{A}f(z, i) = \frac{1}{2} \sigma^2(i) f''(z, i) + \mu(i) f'(z, i) + \sum_{j \in E} q_{ij} \hat{f}(z, j), \quad (z, i) \in \mathcal{E} . \quad (9)$$

This gives the expression for the generator of (Z, J) restricted to the set of bounded functions twice continuously differentiable on the continuous component and satisfying $f'(a(i), i) = f'(b(i), i) = 0$, $i \in E$.

We denote by $\mathcal{E} = \bigcup_{i \in E} [a(i), b(i)] \times \{i\}$ the range of values assumed by the process (Z, J) and define the following subsets of the state space of J ,

$$E^+ = \{j \in E; \sigma(j) > 0 \text{ or } \mu(j) > 0\} , \quad \text{and} \quad E^- = \{j \in E; \sigma(j) > 0 \text{ or } \mu(j) < 0\} .$$

3. Stationary Distribution

If $i \in E^-$, $a(i) \leq Z(0) \leq b(i)$ and $J(0) = i$, then the probability of hitting $a(i)$ before J changes state is bounded below by the positive probability of this event starting from $Z(0) = b(i)$ and $J(0) = i$. This observation together with a geometric retrial argument, recalling that J is irreducible, implies that (Z, J) is a regenerative process with finite

mean regeneration epochs. A similar argument can be made for $i \in E^+$. If $E^- \cup E^+ = \emptyset$ then the process is just a deterministic function of J and thus clearly positive recurrent. Thus, in any case a unique stationary distribution exists.

We show that if a solution to (10) exists then it must be the (unique) stationary distribution. Later we will show how to construct it.

Theorem 1. *The stationary distribution of the process (Z, J) is the unique solution of the following system of differential equations*

$$\frac{1}{2}\sigma^2(i) \Pi_i''(z) - \mu(i) \Pi_i'(z) + \sum_{j \in E} q_{ji} \Pi_j(a(j) \vee z \wedge b(j)) = 0 \quad a(i) \leq z \leq b(i) \quad (10)$$

with boundary conditions $\Pi_i(a(i)) = 0$ and $\Pi_i(b(i)) = \pi_i$, $i \in E$.

PROOF. The stationary distribution satisfies the following equation for any function f belonging to the domain of the generator \mathcal{A} ,

$$\sum_{i \in E} \int_{a(i)}^{b(i)} \mathcal{A}f(z, i) d\Pi_i(z) = 0, \quad (11)$$

Let f be any twice continuous differentiable function on \mathcal{E} with bounded support, then using integration by parts we get that equation (11) reduces to

$$\sum_{i \in E} \int_{a(i)}^{b(i)} \frac{1}{2} \sigma^2(i) \Pi_i''(z) - \mu(i) \Pi_i'(z) df(z, i) + \sum_{i \in E} \int_{a(i)}^{b(i)} \sum_{j \in E} q_{ij} \Pi_i(z) d\hat{f}(z, j) = 0. \quad (12)$$

Noting that

$$\int_{a(i)}^{b(i)} \Pi_i(z) d\hat{f}(z, j) = \int_{a(j)}^{b(j)} \Pi_i(a(i) \vee z \wedge b(i)) df(z, j)$$

and interchanging the indexes i and j in the two sums in the last term of (12) we get that (10) implies (11).

The way of solving the system (10) is to divide the interval $[\min_i \{a(i)\}, \max_i \{b(i)\}]$ to disjoint subintervals, to get a solution of the above system in each of them and then to appropriately glue together all these partial solutions. For this we set $l_0 = \min_i \{a(i)\}$ and by $l_{k+1} = \min_i \{a(i) > l_k\} \wedge \min_i \{b(i) > l_k\}$ and we define the closed intervals $I_k = [l_{k-1}, l_k]$, $k = 1, \dots, K$.

Fix one of these subintervals, say I_k , and let $E_k = \{i \in E : a(i) \leq l_{k-1} < l_k \leq b(i)\}$ be the set of states active over I_k . The restriction of the system (10) to the subinterval I_k then reads as follows

$$\frac{1}{2}\sigma^2(i) \Pi_i''(z) - \mu(i) \Pi_i'(z) + \sum_{j \in E_k} q_{ji} \Pi_j(z) = c_k(i) \quad l_{k-1} \leq z \leq l_k, \quad i \in E_k \quad (13)$$

where we set $c_k(i) = \sum_{j: b(j) \leq l_k} q_{ji} \pi_j$. The equivalent matrix form of (13) is the following

$$\mathbf{S}_k \vec{\Pi}_k''(z) - \mathbf{M}_k \vec{\Pi}_k'(z) + \mathbf{Q}_k^\top \vec{\Pi}_k(z) = \vec{c}_k \quad (14)$$

where $\mathbf{S}_k = \text{diag}[i \in E_k : \sigma^2(i)/2]$, $\mathbf{M}_k = \text{diag}[i \in E_k : \mu(i)]$, $\mathbf{Q}_k = [(i, j \in E_k : q_{ij})]$ and where $\vec{\Pi}_k(z)$ and \vec{c}_k denote the restrictions of the vectors $\vec{\Pi}(z)$ and \vec{c} to the only states active over I_k . $[\cdot]^\top$ is used to denote the transposition operator.

If the set of active states over I_k is a proper subset of E we have that \mathbf{Q} is a strictly substochastic matrix and this has an inverse, therefore the system (14) admits the constant $\vec{k}_k = [\mathbf{Q}^{-1}]^\top \vec{c}$ as particular solution. In the case $E_k = E$, \mathbf{Q} reduces to the rate transition matrix of J that is stochastic and singular. For this case the constant \vec{c} is zero, the system (14) is homogeneous and the particular solution $\vec{k}_k = \vec{0}$ is the zero constant.

Adding to the particular solution the homogeneous solution, we can always write the general solution of (14) in the following form, see [7],

$$\vec{\Pi}_k(z) = \mathbf{\Gamma}_k e^{\mathbf{\Lambda}_k z} \vec{u}_k + \vec{k}_k. \quad (15)$$

where $(\mathbf{\Gamma}_k, \mathbf{\Lambda}_k)$ is a Jordan pair of the matrix polynomial (14) and \vec{u}_k is the unknown vector that can be determined by the boundary conditions. For the specific case of second order matrix polynomial, a Jordan pair consists of a pair of matrix with the following properties

$$\mathbf{S}_k \mathbf{\Gamma}_k \mathbf{\Lambda}_k^2 + \mathbf{M}_k \mathbf{\Gamma}_k \mathbf{\Lambda}_k + \mathbf{Q}_k \mathbf{\Gamma}_k = \mathbb{O}$$

and the rank of $\text{col}[\mathbf{\Gamma}_k, \mathbf{\Gamma}_k \mathbf{\Lambda}_k]$ is maximum.

To be able to glue together the solutions of all intervals I_k , we will see later that it is necessary to be able to solve a system of equations, and this requires to prove that there exists the inverse of the matrix associated to the system.

Let \mathbf{P}_k be the restriction of the matrix \mathbf{P} to the states active on I_k , the main ingredient follows from a slight generalization of the results in [4] where it is shown that the systems

$$\mathbf{S}_k \mathbf{G} \mathbf{L}^2 \mp \mathbf{M}_k \mathbf{G} \mathbf{L} + \mathbf{P}_k^{-1} \mathbf{Q}_k^\top \mathbf{P}_k \mathbf{G} = \mathbb{O} \quad (16)$$

admit solutions $(\mathbf{\Gamma}_k^\pm, \mathbf{\Lambda}_k^\pm)$ that are unique under the restriction that the matrices $\mathbf{\Lambda}_k^\pm$ are substochastic. In this case the matrices $\mathbf{\Gamma}_k^\pm$ are transition matrices and in particular $\mathbf{E}^\pm \mathbf{\Gamma}_k^\pm = \mathbb{I}^\pm$ where the projection matrices \mathbf{E}^\pm are defined as the submatrices of the identity matrix \mathbb{I}_k constructed by selecting only the rows corresponding to the states contained in the set $E_k \cap E^\pm$. The substochastic nature of the matrices $\mathbf{\Lambda}_k^\pm$ is the special characteristic that we are going to use later to prove the non singularity in the final system of equation.

The solutions $(\mathbf{\Gamma}_k^\pm, \mathbf{\Lambda}_k^\pm)$ allow to immediately construct the Jordan Pair in (15) as follows

$$\mathbf{\Gamma}_k = [\mathbf{P}_k \mathbf{\Gamma}_k^+, \mathbf{P}_k \mathbf{\Gamma}_k^-]; \quad \mathbf{\Lambda}_k = \text{diag}[\mathbf{\Lambda}_k^+, -\mathbf{\Lambda}_k^-]. \quad (17)$$

This construction is always valid but in the case when $\mathbf{\Lambda}_k^\pm$ have in common the null eigenvalue. This happens only in the case $E_k = E$ and the asymptotic drift of the modulated process $X(t)$ is zero, that is $\kappa = 0$, see also Section 7 in [4]. We are going to exclude this case as it has no added difficulty if not the one of making more complex all formulas.

Using the special selected Jordan pair in (17), the solution (15) over the interval I_k can be rewritten in the following form

$$\vec{\Pi}_k(z) = \mathbf{P}_k \mathbf{\Gamma}_k^+ e^{+\mathbf{\Lambda}_k^+ z} \vec{u}_k^+ + \mathbf{P}_k \mathbf{\Gamma}_k^- e^{-\mathbf{\Lambda}_k^- z} \vec{u}_k^- + \vec{k}_k, \quad (18)$$

and it is fully specified after assigning the unknown boundary values

$$\Pi_i(l_{k-1}+) \text{ for any } i \in E_k \cap E^+ \quad \text{and} \quad \Pi_i(l_k-) \text{ for any } i \in E_k \cap E^- . \quad (19)$$

To glue together the solution over the entire interval $[l_0, l_K]$ it is then necessary to solve for the unknown boundary conditions in (19) by using the constraints $\Pi_i(a(i)) = 0$ and $\Pi_i(b(i)) = \pi_i$, for any $i \in E$, together with the additional conditions on the continuity of the distribution functions

$$\Pi_i(a(j)-) = \Pi_i(a(j)+) \quad a(i) < a(j) < b(i) \quad (20a)$$

$$\Pi_i(b(j)-) = \Pi_i(b(j)+) \quad a(i) < b(j) < b(i) \quad (20b)$$

for any $i \in E^+ \cup E^-$ and $j \in E$, and on the differentiability of the distribution functions

$$\Pi'_i(a(j)-) = \Pi'_i(a(j)+) \quad a(i) < a(j) < b(i) \quad (21a)$$

$$\Pi'_i(b(j)-) = \Pi'_i(b(j)+) \quad a(i) < b(j) < b(i) \quad (21b)$$

for any $i \in E^+ \cap E^-$ and $j \in E$.

3.1. Computing the stationary distribution

In this section we show how to determine all the unknowns required to get the unique stationary distribution. We are going to consider the sequence of subintervals I_k , with $k = 1, \dots, K$, and we represent the solution $\vec{\Pi}_k(z)$ in the k -th interval according to the form (15). It is worth to say that when applying the model to specific cases it is possible to compute numerically the Jordan Pair in (15) or in its special form (17), like it is shown for example in [13] or in [1]. We used Wolfram Mathematica[®], to get some of the analytical solutions shown in the examples of Sections 4 and 5, but for numerical computation any technical computing system, such as Matlab[®], can be efficiently used for this purpose.

In the same way as we defined the projection matrices E_k^\pm we define the projection matrices for the states that belong to the intersection of the intervals I_{k-1} and I_k , that is the matrices D_k , D_k^+ and \tilde{D}_k . They are defined as the submatrices of the identity matrix \mathbb{I}_k constructed by selecting only the rows corresponding to the states contained respectively in the sets $E_k \cap E_{k-1}$, $(E_k \cap E_{k-1}) \cap E^+$ and $(E_k \cap E_{k-1}) \cap (E^+ \cap E^-)$. For the states in E_k that do not belong to the intersection $E_{k-1} \cap E_k$ we define the additional projection matrices \bar{D}_k and \bar{D}_k^+ defined as the submatrices of the identity matrix \mathbb{I}_k constructed by selecting only the rows corresponding to the states contained respectively in the sets $E_k \cap \bar{E}_{k-1}$, $(E_k \cap \bar{E}_{k-1}) \cap E^+$.

Finally considering the intersection between the intervals I_k and I_{k+1} we define the corresponding projection matrices U_k , U_k^- , \tilde{U}_k , \bar{U}_k and \bar{U}_k^- whose definitions are easy to guess. In all definitions we have assumed that $E_0 = E_{K+1} = \emptyset$.

Applying the continuity and differentiability constraints we get

$$\begin{aligned} \bar{D}_k^+ \Gamma_k e^{l_{k-1} \Lambda_k} \vec{u}_k &= \bar{D}_k^+ \vec{k}_k \\ \bar{U}_k^- \Gamma_k e^{l_k \Lambda_k} \vec{u}_k &= \bar{U}_k^- (\vec{k}_k + \vec{\pi}_k) \\ U_k \Gamma_k e^{l_k \Lambda_k} \vec{u}_k - D_{k+1} \Gamma_{k+1} e^{l_k \Lambda_{k+1}} \vec{u}_{k+1} &= U_k \vec{k}_k - D_{k+1} \vec{k}_{k+1} \\ \tilde{U}_k \Gamma_k \Lambda_k e^{l_k \Lambda_k} \vec{u}_k - \tilde{D}_{k+1} \Gamma_{k+1} \Lambda_{k+1} e^{l_k \Lambda_{k+1}} \vec{u}_{k+1} &= \tilde{U}_k \vec{k}_k - \tilde{D}_{k+1} \vec{k}_{k+1} \end{aligned}$$

for any $k = 1, \dots, K$, with $\vec{\pi}_k = \text{col}[\pi_i, i \in E_k]$, that defining

$$\begin{aligned} \mathbf{A}_k^D &= \text{col}[\bar{\mathbf{D}}_k^+ \mathbf{\Gamma}_k e^{l_{k-1}\mathbf{\Lambda}_k}, \bar{\mathbf{U}}_k^- \mathbf{\Gamma}_k e^{l_k\mathbf{\Lambda}_k}, \mathbf{U}_k \mathbf{\Gamma}_k e^{l_k\mathbf{\Lambda}_k}, \tilde{\mathbf{U}}_k \mathbf{\Gamma}_k \mathbf{\Lambda}_k e^{l_k\mathbf{\Lambda}_k}], \\ \mathbf{A}_k^U &= \text{col}[\mathbb{O}_{k+1}, \mathbb{O}_{k+1}, \mathbf{D}_{k+1} \mathbf{\Gamma}_{k+1} e^{l_k\mathbf{\Lambda}_{k+1}}, \tilde{\mathbf{D}}_{k+1} \mathbf{\Gamma}_{k+1} \mathbf{\Lambda}_{k+1} e^{l_k\mathbf{\Lambda}_{k+1}}], \\ \vec{b}_k &= \text{col}[\bar{\mathbf{D}}_k^+ \vec{k}_k, \bar{\mathbf{U}}_k^- (\vec{k}_k + \vec{\pi}_k), \mathbf{U}_k \vec{k}_k - \mathbf{D}_{k+1} \vec{k}_{k+1}, \tilde{\mathbf{U}}_k \vec{0}_k], \end{aligned}$$

can be rewritten as

$$\mathbf{A}_k^D \vec{u}_k - \mathbf{A}_k^U \vec{u}_{k+1} = \vec{b}_k.$$

Defining the block matrix \mathbf{A} whose block diagonal and upper block diagonal are made respectively of \mathbf{A}_k^D and \mathbf{A}_k^U with $k = 1, \dots, K$, all the unknowns may be solved by resolving the linear system $\mathbf{A}\vec{u} = \vec{b}$, with $\vec{u} = \text{col}[\vec{u}_k, k = 1, \dots, K]$, and $\vec{b} = \text{col}[\vec{b}_k, k = 1, \dots, K]$.

The assumption that $E^+ \cup E^-$ is not empty and that we can choose $i \in E^+ \cup E^-$ such that $a(i) < b(i)$ implies that the matrix \mathbf{A} has dimension at least one. In addition, considering the interval I_k such that $a(i) = l_{k-1}$ if $i \in E^+$ (corr. $b(i) = l_k$ if $i \in E^-$), the corresponding k -block in the matrix \mathbf{A} contains a strictly positive submatrix $\bar{\mathbf{D}}_k^+ \mathbf{\Gamma}_k e^{l_{k-1}\mathbf{\Lambda}_k}$ (resp. $\bar{\mathbf{U}}_k^- \mathbf{\Gamma}_k e^{l_k\mathbf{\Lambda}_k}$). Having that all other submatrices in the same rows are all zeros, we deduce that the matrix \mathbf{A} is invertible if the rank of the k -block column is maximum. It is then easy to realize by induction that \mathbf{A} is invertible if for all $k = 1, \dots, K$ the corresponding k -block column has maximum rank.

Consider the k block, in order to prove that its rank is maximum we can look at a square submatrix with its dimension equals to the size of the block. Noticing that the matrices \mathbf{D}_k^+ (resp. \mathbf{U}_k^-) is a submatrix of \mathbf{D}_k (resp \mathbf{U}_k) that corresponds to the complementary states in $E^+ \cap E_k$ (resp. $E^- \cap E_k$) not selected by $\bar{\mathbf{D}}^+$ (resp. $\bar{\mathbf{U}}^-$), after reordering the rows we get that the sought square submatrix is given by

$$\mathbf{A}_k = \begin{pmatrix} e^{l_{k-1}\mathbf{\Lambda}_k^+} & \mathbf{E}_k^+ \mathbf{P}_k \mathbf{\Gamma}_k^- e^{-l_{k-1}\mathbf{\Lambda}_k^-} \\ \mathbf{E}_k^- \mathbf{P}_k \mathbf{\Gamma}_k^+ e^{l_k\mathbf{\Lambda}_k^+} & e^{-l_k\mathbf{\Lambda}_k^-} \end{pmatrix}.$$

The matrix \mathbf{A}_k is invertible if its Schur complement is. It is given by

$$e^{l_{k-1}\mathbf{\Lambda}_k^+} \left(\mathbb{I}_k - e^{-l_{k-1}\mathbf{\Lambda}_k^+} \mathbf{E}_k^+ \mathbf{P}_k \mathbf{\Gamma}_k^- e^{-l_{k-1}\mathbf{\Lambda}_k^-} e^{-l_k\mathbf{\Lambda}_k^+} \mathbf{E}_k^- \mathbf{P}_k \mathbf{\Gamma}_k^+ e^{-l_k\mathbf{\Lambda}_k^-} \right)$$

and it is invertible by the Levy-Desplanques theorem (see Lemma B.1 in [14]) after noticing that all the matrices in the last term of the equation above are substochastic with at least one strictly substochastic.

4. Fluid queues with modulated buffer

One direct application of the model presented in the previous sections is the case of the fluid queue with Markov modulated buffer. Indeed if we assume that for all states the lower barrier is zero, i.e. $a(i) = 0$ for any $i \in E$, we can look at the process Z as the buffer content of a fluid queue whose net-input flows is the process $X(t)$ defined in (3) and whose buffer is equal to $b(i)$ when the environment J is in state $i \in E$.

In this simplified setting, it is easier to solve the system (10), as it is possible without loss of generality to order the buffer levels in increasing order, i.e. $b(1) \leq b(2) \dots \leq b(N)$.

While the results of previous section are completely general and allow in any case to compute at least numerically the stationary distribution for any configuration of the barriers and the parameters of the Markov modulated Brownian motion, we limit the present discussion only to some few cases where it is possible to write the general solution in an easy form.

It is worth mentioning that a closed form solution can be obtained for the general case when $N = 2$, but the expression is cumbersome and we prefer not to include it. It requires solutions of forth-order algebraic equations, see Section 5.3 for an example of this type but applied to the dividend-payout problem. For this reason we further limit our focus to more specific cases and provide the solution for the case where the net-input flow is not modulated, i.e. the modulation applies only to the buffer levels, and for the case where only in one of the two states the diffusion coefficient is positive.

4.1. Two sided Markov modulated reflection of a (μ, σ) -Brownian motion

In this section we analyze the case when $E = 2$ and the drift and diffusion components, μ and σ , do not depend on the environment. The reflecting levels are given by $a(1) = a(2) = 0$ and $0 < b(1) < b(2)$, and the system can be looked at as a fluid queue that for exponential periods of times, i.e. when $J = 1$, its buffer is reduced to a smaller value. A typical application for this could be a service station that for specified period of times receives help from another service station with which it shares the buffer size. When the second system turns on, the buffer of the first station is reduced and the overflow fluid becomes the starting content for the second station. In the following we derive the stationary distribution of the system, and in the next subsection we compute the discontinuity rate of the content process together with the size distribution of the discontinuities.

The system (10) can be solved by considering the two intervals $I_1 = [0, b(1)]$ and $I_2 = [b(1), b(2)]$. For the interval I_1 we have that

$$\begin{cases} \frac{1}{2}\sigma^2 \Pi_1''(z) - \mu \Pi_1'(z) - q_{12} \Pi_1(z) + q_{21} \Pi_2(z) = 0 \\ \frac{1}{2}\sigma^2 \Pi_2''(z) - \mu \Pi_2'(z) + q_{12} \Pi_1(z) - q_{21} \Pi_2(z) = 0 \end{cases} \quad (22)$$

and for the interval I_2 we have

$$\frac{1}{2}\sigma^2 \Pi_2''(z) - \mu \Pi_2'(z) - q_{21} \Pi_2(z) = -q_{12}\pi_1 \quad (23)$$

In addition we have the boundary conditions $\Pi_i(0) = 0$ and $\Pi_i(b(i)) = \pi_i = q_{3-i,i}/(q_{12} + q_{21})$, $i = 1, 2$, the continuity condition $\Pi_2(b(1)-) = \Pi_2(b(1)+)$ and the differentiability condition $\Pi_2'(b(1)-) = \Pi_2'(b(1)+)$.

Let $\mathbf{\Lambda}_1^\pm = \mathbf{\Theta}_1 \pm \mathbf{\Delta}_1$ be matrix solutions of the equation $\frac{1}{2}\sigma^2 \mathbf{L}^2 \mp \mu \mathbf{L} + \mathbf{Q} = \mathbf{0}$. In this case since $E_1^+ = E_1^- = E$ we have that $\mathbf{\Gamma}_1^+ = \mathbf{\Gamma}_1^- = \mathbb{I}$, in addition $\mathbf{P}_1 = \text{diag}[\pi_1, \pi_2]$. When $\mu < 0$ the matrices $\mathbf{\Lambda}_1^\pm$ below are substochastic and are the same ones appearing in (18). When $\mu > 0$ they are not substochastic but the result still holds. When $\mu = 0$ it follows that the asymptotic drift $\kappa = 0$ and the root 0 has multiplicity 2. As already mentioned before, for this case the solution looks slightly different from the one given below, and it can be computed using similar arguments. We have decided to omit its expression here.

Let λ_1^\pm be the negative solutions of the equations $\frac{1}{2}\sigma^2 z^2 \mp \mu z - (q_{12} + q_{21}) = 0$. Define $\Theta_1 = (\lambda_1^+ + \lambda_1^-)/2 = -(\sqrt{\mu^2 + 2(q_{12} + q_{21})\sigma^2})/\sigma^2$ and $\Delta_1 = (\lambda_1^+ - \lambda_1^-)/2 = \Delta = \mu/\sigma^2$, then we have that

$$\Theta_1 = \Theta_1 \begin{pmatrix} \pi_2 & -\pi_2 \\ -\pi_1 & \pi_1 \end{pmatrix} + \Delta \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix},$$

and $\Delta_1 = \Delta \mathbb{I}$. Considering the continuity conditions at $z = 0$ and $z = b(1)$ we can write the solution for $z \in I_1$ as

$$\begin{pmatrix} \Pi_1(z) \\ \Pi_2(z) \end{pmatrix} = e^{\Delta(z-b(1))} \mathbf{P}_1 \sinh(\Theta_1 z) \sinh^{-1}(\Theta_1 b(1)) \mathbf{P}_1^{-1} \begin{pmatrix} \pi_1 \\ \Pi_2(b(1)) \end{pmatrix}. \quad (24)$$

Let λ_2^\pm be the negative solutions of the equations $\frac{1}{2}\sigma^2 z^2 \mp \mu z - q_{21} = 0$. Define $\Theta_2 = (\lambda_2^+ + \lambda_2^-)/2 = -(\sqrt{\mu^2 + 2q_{21}\sigma^2})/\sigma^2$ and $\Delta_2 = (\lambda_2^+ - \lambda_2^-)/2 = \Delta = \mu/\sigma^2$, we can write the solution for $z \in I_2$ with boundary condition $\Pi_2(b(2)) = \pi_2$ as

$$\Pi_2(z) = \pi_2 - e^{\Delta(z-b(1))} \frac{\sinh(\Theta_2(b(2) - z))}{\sinh(\Theta_2(b(2) - b(1)))} (\pi_2 - \Pi_2(b(1))). \quad (25)$$

In the equation above we already used the continuity condition for $z = b(1)$ by imposing the boundary value at $b(1)$ to be equal to $\Pi_2(b(1))$.

The solution is completely identified by solving for the value of $\Pi_2(b(1))$ that assures the differentiability of $\Pi_2(z)$ at $z = b(1)$.

We have that, for $z \in I_1$

$$\vec{\Pi}'(z) = \Delta \vec{\Pi}(z) + e^{\Delta(z-b(1))} \mathbf{P}_1 \Theta_1 \cosh(\Theta_1 z) \sinh^{-1}(\Theta_1 b(1)) \mathbf{P}_1^{-1} \vec{\Pi}(b(1)),$$

and for $z \in I_2$

$$\Pi_2'(z) = -\Delta (\pi_2 - \Pi_2(z)) + e^{\Delta(z-b(1))} \Theta_2 \frac{\cosh(\Theta_2(b(2) - z))}{\sinh(\Theta_2(b(2) - b(1)))} (\pi_2 - \Pi_2(b(1))).$$

To simplify the resulting expression define $k_i = \vec{e}_2^\top \mathbf{P}_1 \Theta_1 \coth(\Theta_1 b(1)) \mathbf{P}_1^{-1} \vec{e}_i$, $i = 1, 2$, where \vec{e}_i is the i -th column vector of the canonical base in \mathbb{R}^2 , and $k_3 = \Theta_2 \coth(\Theta_2(b(2) - b(1)))$, then the value of $\Pi_2(z)$ at the boundary $b(1)$ is given by

$$\Pi_2(b(1)) = \frac{\pi_2(k_3 - \Delta) - \pi_1 k_1}{k_2 + k_3} \quad (26)$$

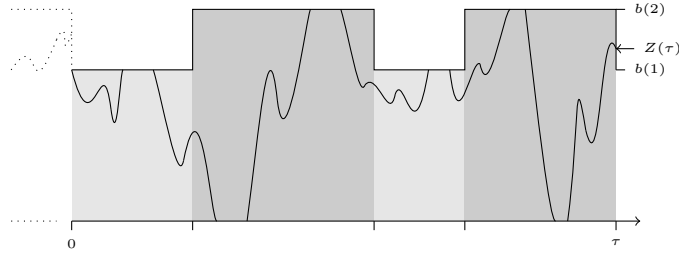
4.1.1. Mean inter-regeneration time and distribution of the content jump

In this section we continue the analysis of the previous section and we are going to study the process only at the moment of the discontinuities, that is when the environment jumps from state 2 to state 1 and the buffer content just before the jump epoch is greater than the buffer level $b(1)$.

These special epochs are regeneration points for the process $Z(t)$, and we denote by τ the general interval of time that lags between two such discontinuities of $Z(t)$. We are going to exploit the regenerative structure of $Z(t)$ at its discontinuity points to determine the expectation of τ and the distribution of $Z(\tau-)$.

These quantities may have relevance when studying an optimization problem. For example, if we consider the example mentioned before, the discontinuity may model the start-up content of a second server to which there can be associated start-up costs taken into account by a given cost function. The value of $1/\mathbb{E}[\tau]$ gives the average rate at which such costs incur.

Assume that $Z(0-) > b(1)$ and $Z(0) = b(1)$ and define $\tau = \inf_{t>0} \{\Delta Z(t) < 0\}$, the picture below helps in understanding the given definition.



Let $f(z)$ be any twice differentiable function, by Itô's Lemma we have that

$$\begin{aligned} f(Z(t)) = f(Z(0)) &+ \sigma \int_0^t f'(Z(s)) dW(s) + \int_0^t \frac{1}{2} \sigma^2 f''(Z(s)) + \mu f'(Z(s)) ds \\ &+ f'(0) L(t) - f'(b(1)) U_1(t) - f'(b(2)) U_2(t) \end{aligned} \quad (27)$$

where $U_i(t)$, $i \in E$ is the amount of content lost at the barrier $b(i)$ during the interval $[0, t]$. Evaluating (27) at $t = \tau$ under the condition that $Z(0) = b(1)$ and taking expectation we get that

$$\begin{aligned} \mathbb{E}_{b(1)}[f(Z(\tau)) - f(b(1))] &= \mathbb{E}_{b(1)} \left[\int_0^\tau \frac{1}{2} \sigma^2 f''(Z(s)) + \mu f'(Z(s)) ds \right] \\ &+ f'(0) \mathbb{E}_{b(1)}[L(\tau)] - f'(b(1)) \mathbb{E}_{b(1)}[U_1(\tau)] \\ &- f'(b(2)) \mathbb{E}_{b(1)}[U_2(\tau)] \end{aligned} \quad (28)$$

Having that τ is a regeneration point for Z we have by renewal theory that

$$\Pi(z) = \mathbb{P}\{Z^* \leq z\} = \eta \mathbb{E}_{b(1)} \left[\int_0^\tau 1_{\{Z(s) \leq z\}} ds \right],$$

where we defined $\eta = 1/\mathbb{E}_{b(1)}[\tau]$, the rate of discontinuities of Z .

In general we have that

$$\mathbb{E}[f(Z^*)] = \int_0^{b(2)} f(z) \Pi(dz) = \eta \mathbb{E}_{b(1)} \left[\int_0^\tau f(Z(s)) ds \right], \quad (29)$$

so that multiplying equation (28) by η , applying (29) and using repeatedly integration by parts similarly to the proof of Theorem 1, we get the following system of differential

equations by collecting all the integrand terms with factor $f(z)$,

$$\frac{1}{2}\sigma^2 \pi''(z) - \mu \pi'(z) = \eta H'(z) 1_{\{b(1) < z \leq b(2)\}} \quad (30)$$

where $H(z) = \mathbb{P}\{b(1) \leq Z(\tau-) \leq z\}$, with $b(1) < z \leq b(2)$, and $\pi(z) = \Pi'(z)$.

Having that $\Pi(z) = \Pi_2(z) + \Pi_1(z \wedge b(1))$, by comparing the second equation in (30) with the derivative of equation (23) we get that

$$\eta H'(z) = q_{21} \Pi'_2(z) . \quad (31)$$

Integrating last equation over the interval $[b(1), b(2)]$ with boundary conditions $H(b(1)) = 0$ and $H(b(2)) = 1$ it follows that $\eta = q_{21} (\pi_2 - \Pi_2(b(1)))$. Substituting its value in (31) and integrating we get that, for $z \in I_2$,

$$H(z) = \frac{\Pi_2(z) - \Pi_2(b(1))}{\pi_2 - \Pi_2(b(1))} = 1 - e^{\Delta(z-b(1))} \frac{\sinh(\Theta_2(b(2) - z))}{\sinh(\Theta_2(b(2) - b(1)))} ,$$

where we used the expression of $\Pi_2(z)$ given in (25).

4.2. Two-state modulation with at least one with no diffusion component

In this section we consider the case when $N = 2$ and only in one of the two states the diffusion coefficient is positive.

The way to proceed to compute the stationary distribution is similar to the one used in the previous section, but we decided to include these examples because for these cases the matrices Γ^\pm are rectangular and do not reduce to the identity matrix. In [4] and [5] the authors looked at how compute the stationary distribution for the case of two-side reflection with two non-modulated barriers, but no explicit examples were given and as far as we know they are not treated elsewhere.

The system (10) can be solved by considering the two intervals $I_1 = [0, b(1)]$ and $I_2 = [b(1), b(2)]$. We consider the two cases when the state with no diffusion component is the first one, i.e. $\sigma_1 = 0$ and then when the state with no diffusion is the second one, i.e. $\sigma_2 = 0$.

In the second case in order to have positive probability for the process to enter the interval $(b(0), b(1)]$ we assume that $\mu_2 > 0$. To simplify the analysis and reduce the cases we assume that the asymptotic drift $\kappa < 0$ and for the first case that $\mu_1 < 0$.

Case $\mu_1 < 0, \sigma_1 = 0; \sigma_2 > 0; \kappa < 0$: For $z \in I_1$ we have that

$$\begin{cases} -\mu_1 \Pi'_1(z) - q_{12} \Pi_1(z) + q_{21} \Pi_2(z) = 0 \\ \frac{1}{2}\sigma_2^2 \Pi''_2(z) - \mu_2 \Pi'_2(z) + q_{12} \Pi_1(z) - q_{21} \Pi_2(z) = 0 \end{cases}$$

and for $z \in I_2$ we have

$$\frac{1}{2}\sigma_2^2 \Pi''_2(z) - \mu_2 \Pi'_2(z) - q_{21} \Pi_2(z) = -q_{12}\pi_1 .$$

In addition we have the boundary conditions $\Pi_i(0) = 0$ and $\Pi_i(b(i)) = \pi_i = q_{3-i,i}/(q_{12} + q_{21})$, $i = 1, 2$, the continuity condition $\Pi_2(b(1)-) = \Pi_2(b(1)+)$ and the differentiability condition $\Pi'_2(b(1)-) = \Pi'_2(b(1)+)$.

Having $\mu_1 < 0$ it follows that $E^+ = \{2\}$ and $E^- = \{1, 2\}$. Hence defining $\gamma_1^+ = \mathbb{P}_1\{\sup_{t>0}\{X(t)\} = 0\}$ we have that $\mathbf{\Gamma}_1^+ = (\gamma_1^+, 1)^\top$ and $\mathbf{\Gamma}_1^- = \mathbb{I}^-$.

Solving the system (16) that is explicitly rewritten as

$$\begin{cases} q_{12}(1 - \gamma_1^+) - \mu_1 \lambda_1^+ \gamma_1^+ & = 0 \\ -q_{21}(1 - \gamma_1^+) - \mu_2 \lambda_1^+ + \frac{1}{2}\sigma_2^2 (\lambda_1^+)^2 & = 0 \end{cases}$$

and selecting the solution with $\lambda_1^+ < 0$ we have that

$$\lambda_1^+ = \frac{\mu_2}{\sigma_2^2} - \frac{q_{12}}{\mu_1} - \frac{1}{\mu_1 \sigma_2^2} \delta_1; \quad \gamma_1^+ = -\frac{1}{2} \frac{\mu_2}{\mu_1} \frac{q_{12}}{q_{21}} - \frac{q_{12}^2 \sigma_2^2}{4 q_{21} \mu_1^2} - \frac{1}{2 \mu_1^2} \frac{q_{12}}{q_{21}} \delta_1$$

where $\delta_1 = \sqrt{\mu_1^2 \mu_2^2 + 2 \mu_1 \mu_2 q_{12} \frac{\sigma_2^2}{2} + 4 \mu_1^2 q_{21} \frac{\sigma_2^2}{2} + q_{12}^2 \frac{\sigma_2^4}{4}}$. Solving the system (16) that is explicitly rewritten as

$$\begin{cases} q_{12} + \mu_1 \lambda_{12}^- & = 0 \\ q_{21} + \mu_2 \lambda_{21}^- - \frac{1}{2}\sigma_2^2 \lambda_{12}^- \lambda_{21}^- - \frac{1}{2}\sigma_2^2 (\lambda_{21}^-)^2 & = 0 \end{cases}$$

we get the following solutions

$$\lambda_{12}^- = -\frac{q_{12}}{\mu_1}; \quad \lambda_{21}^- = \frac{\mu_2}{\sigma_2^2} + \frac{q_{12}}{2\mu_1} + \frac{1}{\mu_1 \sigma_2^2} \delta_2$$

where $\delta_2 = \sqrt{2 q_{21} \mu_1^2 \sigma_2^2 + [\mu_1 \mu_2 + q_{12} \frac{1}{2} \sigma_2^2]^2}$.

Over the interval I_1 , the stationary distribution is given by

$$\begin{pmatrix} \Pi_1(z) \\ \Pi_2(z) \end{pmatrix} = \mathbf{P}_1 \mathbf{\Gamma}_1^+ e^{z \lambda_1^+} c_1^+ + \mathbf{P}_1 \mathbf{\Gamma}_1^- e^{-z \Lambda_1^-} \bar{c}_1^-.$$

Having $\Pi_2(0) = 0$ we can solve for $c_1^+ = -\bar{c}_1^-$ and with $\Pi_1(b(1)) = \pi_1$ we have

$$\begin{pmatrix} \Pi_1(z) \\ \Pi_2(z) \end{pmatrix} = \mathbf{P}_1 \left[\hat{\mathbf{\Gamma}}_1^+ e^{z \Lambda_1^+} - \mathbf{\Gamma}_1^- e^{-z \Lambda_1^-} \right] \mathbf{C}^{-1} \mathbf{P}_1^{-1} \begin{pmatrix} \pi_1 \\ \Pi_2(b(1)) \end{pmatrix}, \quad (32)$$

where $\hat{\mathbf{\Gamma}}_1^+ = \begin{pmatrix} 0 & \gamma_1^+ \\ 0 & 1 \end{pmatrix}$, $\Lambda_1^+ = \lambda_1^+ \mathbb{I}$ and $\mathbf{C} = [\hat{\mathbf{\Gamma}}_1^+ e^{b(1) \Lambda_1^+} - \mathbf{\Gamma}_1^- e^{-b(1) \Lambda_1^-}]$.

Over the interval I_2 let λ_2^\pm be the negative solutions of the equations $\frac{1}{2}\sigma_2^2 z^2 \mp \mu_2 z - q_{21} = 0$. Define $\Theta_2 = (\lambda_2^+ + \lambda_2^-)/2 = -(\sqrt{\mu_2^2 + 2 q_{21} \sigma_2^2})/\sigma_2^2$ and $\Delta_2 = (\lambda_2^+ - \lambda_2^-)/2 = \mu_2/\sigma_2^2$, we can write the solution for $z \in I_2$ with boundary condition $\Pi_2(b(2)) = \pi_2$ as

$$\Pi_2(z) = \pi_2 - e^{\Delta_2(z-b(1))} \frac{\sinh(\Theta_2(b(2)-z))}{\sinh(\Theta_2(b(2)-b(1)))} (\pi_2 - \Pi_2(b(1))). \quad (33)$$

In the equation above we already used the continuity condition for $z = b(1)$ by imposing the boundary value at $b(1)$ to be equal to $\Pi_2(b(1))$.

The solution is completely identified by solving for the value of $\Pi_2(b(1))$ that assures the differentiability of $\Pi_2(z)$ at $z = b(1)$.

We have that for $z \in I_1$

$$\vec{\Pi}'(z) = \mathbf{P}_1 \left[\hat{\mathbf{\Gamma}}_1^+ \mathbf{\Lambda}_1^+ e^{z \mathbf{\Lambda}_1^+} + \mathbf{\Gamma}_1^- \mathbf{\Lambda}_1^- e^{-z \mathbf{\Lambda}_1^-} \right] \mathbf{C}^{-1} \mathbf{P}_1^{-1} \begin{pmatrix} \pi_1 \\ \Pi_2(b(1)) \end{pmatrix}$$

and for $z \in I_2$

$$\Pi_2'(z) = -\Delta_2 (\pi_2 - \Pi_2(z)) + e^{\Delta_2(z-b(1))} \Theta_2 \frac{\cosh(\Theta_2(b(2) - z))}{\sinh(\Theta_2(b(2) - b(1)))} (\pi_2 - \Pi_2(b(1))) .$$

To simplify the resulting expression define

$$k_i = \vec{e}_i^\top \mathbf{P}_1 \left[\hat{\mathbf{\Gamma}}_1^+ \mathbf{\Lambda}_1^+ e^{b(1) \mathbf{\Lambda}_1^+} + \mathbf{\Gamma}_1^- \mathbf{\Lambda}_1^- e^{-b(1) \mathbf{\Lambda}_1^-} \right] \mathbf{C}^{-1} \mathbf{P}_1^{-1} \vec{e}_i, \quad i = 1, 2 ,$$

where \vec{e}_i is the i -th column vector of the canonical base in \mathbb{R}^2 , and

$$k_3 = \Theta_2 \coth(\Theta_2(b(2) - b(1))) ,$$

then the value of $\Pi_2(z)$ at the boundary $b(1)$ is given by

$$\Pi_2(b(1)) = \frac{\pi_2(k_3 - \Delta_2) - \pi_1 k_1}{k_2 + k_3} . \quad (34)$$

Notice that in this case the function $\Pi_1(z)$ is not continuous at $z = 0$ where it gives the probability to find the system empty when the environment is found in state 1, i.e.

$$\mathbb{P}(Z^* = 0, J^* = 1) = \Pi_1(0) = \vec{e}_1^\top \mathbf{P}_1 \left[\hat{\mathbf{\Gamma}}_1^+ - \mathbf{\Gamma}_1^- \right] \mathbf{C}^{-1} \mathbf{P}_1^{-1} \begin{pmatrix} \pi_1 \\ \Pi_2(b(1)) \end{pmatrix} .$$

Case $\sigma_1 > 0$; $\mu_2 > 0$, $\sigma_2 = 0$; $\kappa < 0$: For the interval I_1 we have that

$$\begin{cases} \frac{1}{2} \sigma_1^2 \Pi_1''(z) - \mu_1 \Pi_1'(z) - q_{12} \Pi_1(z) + q_{21} \Pi_2(z) = 0 \\ -\mu_2 \Pi_2'(z) + q_{12} \Pi_1(z) - q_{21} \Pi_2(z) = 0 \end{cases}$$

and for the interval I_2 we have

$$-\mu_2 \Pi_2'(z) - q_{21} \Pi_2(z) = -q_{12} \pi_1 .$$

In addition we have the boundary conditions $\Pi_i(0) = 0$ and $\Pi_i(b(i)) = \pi_i = q_{3-i,i}/(q_{12} + q_{21})$, $i = 1, 2$ and the continuity condition $\Pi_2(b(1)-) = \Pi_2(b(1)+)$.

Having $\mu_2 > 0$ it follows that $E^+ = \{1, 2\}$ and $E^- = \{1\}$. In a similar way as in the previous case we have $\mathbf{\Gamma}_1^+ = \mathbb{I}^+$ and $\mathbf{\Gamma}_1^- = (1, \gamma_2^-)^\top$ where $\gamma_2^- = \mathbb{P}_2\{\inf_{t>0}\{X(t)\} = 0\}$. Solving the system (16) that is explicitly rewritten as

$$\begin{cases} q_{12} - \mu_1 \lambda_{12}^+ - \frac{1}{2} \sigma_1^2 \lambda_{21}^+ \lambda_{12}^+ - \frac{1}{2} \sigma_1^2 (\lambda_{12}^+)^2 = 0 \\ q_{21} - \mu_2 \lambda_{21}^+ = 0 \end{cases}$$

we get the following solutions

$$\lambda_{12}^+ = -\frac{\mu_1}{\sigma_1^2} - \frac{q_{21}}{2\mu_2} + \frac{1}{\mu_2 \sigma_1^2} \delta_1; \quad \lambda_{21}^+ = \frac{q_{21}}{\mu_2}$$

where $\delta_1 = \sqrt{2q_{12} \mu_2^2 \sigma_1^2 + [\mu_2 \mu_1 + q_{21} \frac{1}{2} \sigma_1^2]^2}$.

Solving the system (16) that is explicitly rewritten as

$$\begin{cases} -q_{21}(1 - \gamma_2^-) + \mu_1 \lambda_1^- + \frac{1}{2} \sigma_1^2 (\lambda_1^-)^2 &= 0 \\ q_{21}(1 - \gamma_2^-) + \mu_2 \lambda_1^- \gamma_2^- &= 0 \end{cases}$$

and selecting the solution with $\lambda_1^- < 0$ we have that

$$\lambda_1^- = \frac{-\mu_1}{\sigma_1^2} + \frac{q_{21}}{\mu_2} - \frac{1}{\mu_2 \sigma_1^2} \delta_2; \quad \gamma_2^- = -\frac{1}{2} \frac{\mu_1}{\mu_2} \frac{q_{21}}{q_{12}} - \frac{q_{21}^2 \sigma_1^2}{4 q_{12} \mu_2^2} + \frac{1}{2 \mu_2^2} \frac{q_{21}}{q_{12}} \delta_2$$

where $\delta_2 = \sqrt{\mu_2^2 \mu_1^2 + 2 \mu_2 \mu_1 q_{21} \frac{\sigma_1^2}{2} + 4 \mu_2^2 q_{12} \frac{\sigma_1^2}{2} + q_{21}^2 \frac{\sigma_1^4}{4}}$.

Over the interval I_1 , the stationary distribution is given by

$$\begin{pmatrix} \Pi_1(z) \\ \Pi_2(z) \end{pmatrix} = \mathbf{P}_1 \mathbf{\Gamma}_1^+ e^{z \mathbf{\Lambda}_1^+} \vec{c}_1^+ + \mathbf{P}_1 \mathbf{\Gamma}_1^- e^{-z \lambda_1^-} c_1^-.$$

Having $\vec{\Pi}(0) = \vec{0}$ we can solve for $\vec{c}_1^+ = -c_1^- \mathbf{\Gamma}_1^-$ and knowing that $\Pi_1(b(1)) = \pi_1$ we get

$$\begin{pmatrix} \Pi_1(z) \\ \Pi_2(z) \end{pmatrix} = \frac{\pi_1}{c_1} \mathbf{P}_1 (e^{-z \mathbf{\Lambda}_1^-} - e^{z \mathbf{\Lambda}_1^+}) \mathbf{\Gamma}_1^- \quad (35)$$

where $\mathbf{\Lambda}_1^- = \lambda_1^- \mathbb{I}$ and $c_1 = \vec{c}_1^\top \mathbf{P}_1 (e^{-b(1) \mathbf{\Lambda}_1^-} - e^{b(1) \mathbf{\Lambda}_1^+}) \mathbf{\Gamma}_1^-$.

Over the interval I_2 the solution is simply

$$\Pi_2(z) = \pi_2 - e^{-\frac{q_{21}}{\mu_2}(z-b(1))} (\pi_2 - \Pi_2(b(1))), \quad (36)$$

where $\Pi_2(b(1))$ is known by continuity from equation (35) and we used the fact that $\pi_2 = \pi_1 q_{12}/q_{21}$.

Notice that in this case the function $\Pi_2(z)$ is not continuous at $z = b(2)$ where it gives the probability to find the system saturated when the environment is found in state 2, i.e.

$$\mathbb{P}(Z^* = b(2), J^* = 2) = \pi_2 - \Pi_2(b(2-)) = e^{-\frac{q_{21}}{\mu_2}(z-b(1))} (\pi_2 - \Pi_2(b(1))).$$

5. Synchronized Barrier Strategies for Dividend Payout

In this section we look at an application of the model presented in Section 2 to the problem of computing the expected dividend payouts of a company. We assume that the company profit fluctuates according to the Markov modulated Brownian motion

$(X(t), J(t))$ where $X(t)$ is as in (3) and $J(t)$ is the environment process as introduced in Section 2.

The model is a direct extension of the one studied in [6] where for the profit process it was chosen a free Brownian motion. the introduction of the external environment $J(t)$ from a modeling perspective helps in adapting the process to the case where the profit process may depend on external environmental situation, such as for example seasonal-dependent activities.

We assume that the company select for each state of the environment $J(t)$ a barrier level $b(J(t))$ and decides to pay dividends as soon as its surplus process reaches that level. In this way the surplus process behaves exactly as the Markov Modulated two-sided reflected Brownian motion $Z(t)$ up to the moment of the first hitting time of the lower barrier, i.e. $\tau = \inf\{t \geq 0 : Z(t) = 0\}$, that corresponds to the ruin time for the company.

The non-discounted total dividends paid up to time τ is directly given by $U(\tau)$, that is the upper regulator process computed at the ruin epoch. In general, given the discount rate $\delta > 0$, this amount is given by

$$U = \int_0^\tau e^{-\delta t} U(dt) . \quad (37)$$

U is a random quantity that depends on the path realization of the process (X, J) , as well as the selected barriers, $\{b(j)\}_{j \in E}$, and the start-up condition of the company $(z, j) = (Z(0), J(0))$. The aim of the company is to compute to then optimize the expected value of total discount dividend paid over its time-horizon. We denote this quantity by $V(z, j)$, when then starts at time 0 with initial capital z and in an environment state j . The formal definition is given by

$$V(z, j) = \mathbb{E}[U | Z(0) = z, J(0) = j] = \mathbb{E}_{(z, j)}[U] . \quad (38)$$

In the following we show how to heuristically determine the system of differential equation that admits as solution the desired quantity $V(z, j)$, in the next sections we show how to get the same system of equations in a more rigorous way.

Assuming that the starting position z is far from the barrier level $b(j)$ we can assume that in a relatively short interval Δt the surplus process does not reach the reflecting barrier such that the following relation holds

$$\mathbb{E}_{(z, j)} \left[V(Z(\Delta t), J(\Delta t)) \right] = e^{\delta \Delta t} V(z, j) - e^{\delta \Delta t} \mathbb{E}_{(z, j)} \left[\int_0^{\Delta t} e^{-\delta t} U(dt) \right] .$$

In addition we have that

$$\mathbb{E}_{(z, j)} \left[\int_0^{\Delta t} e^{-\delta t} U(dt) \right] = \sum_{k \in E} q_{jk} (z - b_k)^+ \Delta t + o(\Delta t)$$

so that using the first order Taylor expansion of $e^{\delta \Delta t}$, we finally get

$$\mathbb{E}_{(z, j)} \left[V(Z(\Delta t), J(\Delta t)) \right] = V(z, j) + \delta V(z, j) \Delta t - \sum_{k \in E} q_{jk} (z - b_j)^+ \Delta t + o(\Delta t) .$$

On the other side using the Itô formula it follows

$$\begin{aligned} V(Z(\Delta t), J(\Delta t)) &= V(Z(0), J(0)) + \mu(J(0)) V'(Z(0), J(0)) \Delta t \\ &\quad + \frac{1}{2} \sigma^2(J(0)) V''(Z(0), J(0)) \Delta t \\ &\quad + \sum_{k \in E} q_{J(0)k} V(Z(0) \wedge b(k), k) \Delta t + o(\Delta t) \end{aligned}$$

and equating the two expressions above we get the following differential equations

$$\frac{1}{2} \sigma^2(j) V''(z, j) + \mu(j) V'(z, j) - \delta V(z, j) + \sum_k q_{jk} [V(z \wedge b(k), k) + (z - b(k))^+] = 0. \quad (39)$$

5.1. Regularity of $V(z, j)$

The derivation of equation (39) has been done by implicitly assuming regularity condition of the unknown function $V(z, j)$, i.e. that it has second derivative on the interval $(0, b(j))$ with the exception of at most isolated points. In this section we prove that indeed $V(z, j)$ does admits second derivative and again that it satisfies equation (39). In the following we assume that $\sigma(j) > 0$ as, according to (39), in the case it was zero we would need only the first derivative of $V(z, j)$. The treatment below can be easily adapted to handle this case.

Define $T_b = \min_{j \in E} \{\inf_{t \geq 0} \{(Z(t), J(t)) = (b(j), j)\}\}$ and $T_0 = \inf\{t \geq 0, X(t) = 0\}$ we have that

$$V(z, j) = \mathbb{E}_{(z, j)}[e^{-\delta T_b} (V(b(J(T_b)), J(T_b)) - \Delta X(T_b)) 1\{T_b < T_0\}]$$

with $\Delta X(t) = X(t) - X(t-)$.

Define the stopping time $\tau(h) = T_{z-h} \wedge T_{z+h} \wedge T_J$ with $T_J = \inf\{t > 0, J(t-) \neq J(t)\}$ and $T_{z \pm h} = \inf\{t > 0, X(t) = z \pm h\}$, we have that

$$\begin{aligned} V(z, j) &= \mathbb{E}_{(z, j)}[e^{-\delta \tau(h)} (V(X(\tau(h)), J(\tau(h))) + \Delta X(\tau(h)))] \\ &= g_+(h) V(z + h, j) + g_-(h) V(z - h, j) + \sum_{k \in E} \tilde{V}_J(h, k) \frac{q_{jk}}{q_j} g_J(h) \end{aligned} \quad (40)$$

where $g_J(h) = \mathbb{P}_{(z, j)}\{\tau(h) = T_J\}$, $g_{\pm}(h) = \mathbb{E}_{(z, j)}[e^{-\delta \tau(h)}; \tau(h) = T_{z \pm h}]$, $q_j = \sum_{k \neq j} q_{jk}$, and

$$\tilde{V}_J(h, k) = \mathbb{E}_{(z, j)}[e^{-\delta T_J} (V(X(T_J), k) + \Delta X(T_J)) | \tau(h) = T_J, J(T_J) = k].$$

To compute $g_J(h)$ we have that

$$g_J(h) = \mathbb{P}_{(z, j)}\left\{\sup_{0 \leq t \leq T_J} |X(t) - z| < h\right\} = \mathbb{P}_0\{T_{|h|} > T_J\} = 1 - \mathbb{E}_0[e^{-q_J T_{|h|}}],$$

where $T_{|h|} = \inf_{t \geq 0} \{|Y(t)| = h\}$ is the hitting time of the set $(-h, h)^c$ of the process $Y(t)$ that is a Brownian motion with drift $\mu(j)$ and diffusion coefficient $\sigma(j)$. It is known, see [2], that

$$\mathbb{E}_0[e^{-\lambda T_{|h|}}] = \cosh\left(\frac{h\mu}{\sigma}\right) \operatorname{sech}\left(\frac{h\mu\sqrt{2\lambda + \mu^2}}{\sigma}\right) = 1 - \lambda h^2 + o(h^2), \quad (41)$$

and from this it follows that $g_J(h) = 1 - \mathbb{E}_0[e^{-q_j T_{|h|}}] = \frac{q_j}{\sigma^2} h^2 + o(h^2)$.

To compute $g_{\pm}(h)$ we have that

$$\begin{aligned} g_{\pm}(h) &= \mathbb{E}_{(z,j)}[\mathbb{E}[e^{-\delta \tau(h)}; \tau(h) = T_{z \pm h} | T_J]] \\ &= \int_{t=0}^{\infty} \mathbb{E}[e^{-\delta \tau(h)}; \tau(h) = T_{z \pm h} | T_J = t] q_j e^{-q_j t} dt \\ &= \int_{t=0}^{\infty} \mathbb{E}[e^{-\delta \tau(h)} \mathbb{E}[1\{\tau(h) = T_{z \pm h}\} | \tau(h), T_J = t] | T_J = t] q_j e^{-q_j t} dt \end{aligned}$$

and since the exit location $X(T_{z+h} \wedge T_{z-h})$ is independent of the exit time $T_{z+h} \wedge T_{z-h}$, see [15], assuming $\mu > 0$, we have that

$$\mathbb{E}[1\{\tau(h) = T_{z \pm h}\} | \tau(h), T_J = t] = c_{\pm} 1\{\tau(h) < t\}$$

where $c_{\pm} = \frac{1}{2} \pm \frac{1}{2}(\exp(\frac{2\mu}{\sigma^2} h) - 1)/(\exp(\frac{2\mu}{\sigma^2} h) + 1)$, therefore

$$\begin{aligned} g_{\pm}(h) &= c_{\pm} \int_{t=0}^{\infty} \mathbb{E}[e^{-\delta \tau(h)} 1\{\tau(h) < t\} | T_J = t] q_j e^{-q_j t} dt \\ &= c_{\pm} \int_{t=0}^{\infty} \int_{x=0}^t e^{-\delta x} \mathbb{P}\{T_{|h|} \in dx\} q_j e^{-q_j t} dt = c_{\pm} \mathbb{E}_0[e^{-(\delta+q_j) T_{|h|}}] \\ &= c_{\pm} \cosh\left(\frac{2\mu h}{\sigma}\right) \operatorname{sech}\left(\frac{2\mu}{\sigma} h \sqrt{\frac{(\delta+q_j)}{2\mu^2} + \frac{1}{4}}\right) \end{aligned}$$

Last term in (40) is positive, it follows that

$$\begin{aligned} V(z, j) &\leq g_+(h) V(z+h, j) + g_-(h) V(z-h, j) \\ &\leq G_+(h) V(z+h, j) + G_-(h) V(z-h, j) \end{aligned}$$

where $G_{\pm}(h) = g_{\pm}(h)/(g_+(h) + g_-(h))$. Since $G_+(h) + G_-(h) = 1$ and $G_-(h) \leq \frac{1}{2} \leq G_+(h)$ the follow inequality holds

$$V(z, j) - V(z+h, j) \leq V(z-h, j) - V(z, j)$$

that implies that $V(z, j)$ is continuous and concave in $(0, b(j))$, see Courant [3, Ch. IV.2 page. 326].

Rearranging terms in (40) and using that $1 - (g_+(h) + g_-(h)) = o(h)$ we have

$$g_-(h)[V(z, j) - V(z-h, j)] = g_+(h)[V(z+h, j) - V(z, j)] + o(h),$$

and dividing it by h , letting $h \rightarrow 0$ and having $g_{\pm}(h) \rightarrow 1/2$ as $h \rightarrow 0$ we finally get

$$V'(z-, j) = V'(z+, j),$$

i.e. the function $V(z, j)$ is differentiable in $(0, b_j)$. Define

$$\lim_{h \rightarrow 0} \frac{V(z+h, j) - 2V(z, j) + V(z-h, j)}{h^2} = \psi(z)$$

with

$$\begin{aligned} \psi(z, j) = \lim_{h \rightarrow 0} & \left[\frac{2g_+(h) - 1}{h} \frac{V(z + h, j) - V(z, j)}{h} \right. \\ & + \frac{1 - 2g_-(h)}{h} \frac{V(z, j) - V(z - h, j)}{h} + \frac{2g_-(h) + 2g_-(h) - 2}{h^2} V(z, j) \\ & \left. + \frac{h^2 \sum_k q_{jk} [V(z \wedge b_k, k) + (z - b_k)^+] + o(h^2)}{h^2} \right], \end{aligned}$$

where we used the fact that $\tilde{V}_J(h, k) \rightarrow V(z \wedge b_k, k) + (z - b_k)^+$ as $h \rightarrow 0$ by bounded convergence. Having that $(2g_\pm(h) - 1)/h \rightarrow \pm\mu/\sigma^2$ and $(2g_-(h) + 2g_-(h) - 2)/h^2 \rightarrow 2(\delta + q_j)/\sigma^2$ as $h \rightarrow 0$ we get that

$$\psi(z, j) = \frac{2\mu}{\sigma^2} V'(z, j) - \frac{2(\delta + q_j)}{\sigma^2} V(z, j) + \sum_{k \neq j} q_{jk} [V(z \wedge b_k, k) + (z - b_k)^+]. \quad (42)$$

Using Schwarz's Theorem, [16], we get that $V(z, j)$ is twice differentiable and $V''(z, j) = \psi(z, j)$. In addition (42) coincides with (39).

5.2. Boundary conditions

To determine the value of $V(z, j)$ is necessary to add to the differential equations (39) two boundary conditions for each $j \in E$, at the barriers 0 and $b(j)$.

It is obvious that $V(0, j) = 0$, for any $j \in E$ and in addition we have

$$V'(b(j), j) = 1.$$

This equation that can be found for the non-modulated case in [6] has the following explanation.

Assume Δz is small, starting at $b(j) - \Delta z$ we will touch the barrier $b(j)$ at time $T_{\Delta z}$ so that $e^{-\delta T_{\Delta z}} = 1 + o(T_{\Delta z})$. Defining $\tau = T_J \wedge T_0$, the shortest time between a change of state for J or the ruin epoch, we have that the process $Z(t)$ will have the same dynamic of a single reflected $(\mu(j), \sigma(j))$ -Brownian motion at the barrier $b(j)$ for $0 \leq t \leq \tau$ that we denote by $Y(t)$. The upper regulator process is known to have the following expression

$$U_z(t) = z + \sup\{0 \leq s \leq t : Y(s) \vee b(j) - z\}$$

as $0 \leq t \leq \tau$ when $Z(0) = z$, notice that by definition $Y(0) = 0$. Since $\int_0^t e^{-\delta s} U(ds) = U(t) + o(t)$, it follows that

$$V(b(j), j) - V(b(j) - \Delta z, j) = \Delta z + \mathbb{E}[1\{M(t) < \Delta z\}(M(t) - \Delta z)] + o(\mathbb{E}[T_{\Delta z}]),$$

where $M(t) = \sup\{0 \leq s \leq t : Y(s)\}$ and $T_{\Delta z} = \inf\{t > 0, M(t) \geq \Delta z\}$. Having that

$$-\Delta z \mathbb{P}(T_{\Delta z} < \tau) \leq \mathbb{E}[1\{M(t) < \Delta z\}(M(t) - \Delta z)] \leq 0$$

we can get the following bounds

$$1 - \mathbb{P}(T_{\Delta z} > \tau) + \frac{o(\mathbb{E}[T_{\Delta z}])}{\Delta z} \leq \frac{V(b(j), j) - V(b(j) - \Delta z, j)}{\Delta z} \leq 1 + \frac{o(\mathbb{E}[T_{\Delta z}])}{\Delta z},$$

and taking the limit for $\Delta z \rightarrow 0$ and using the fact that $\mathbb{P}(T_{\Delta z} > \tau) \rightarrow 0$ and $o(\mathbb{E}[T_{\Delta z}])/\Delta z \rightarrow 0$ we obtain the result.

5.3. Expected dividend payout - Case $|E| = 2$

Assuming $q_{12} = q_{21} = \lambda$, we have that for $z \in I_1$ the system of differential equations (39) reduces to

$$\begin{cases} \frac{1}{2}\sigma^2(1)V''(z,1) + \mu(1)V'_1(z,1) - (\lambda + \delta)V(z,1) + \lambda V(z,2) = 0 \\ \frac{1}{2}\sigma^2(2)V''(z,2) + \mu(2)V'_1(z,2) + \lambda V(z,1) - (\lambda + \delta)V(z,2) = 0 \end{cases}$$

and for $z \in I_2$ it reduces to

$$\frac{1}{2}\sigma^2(2)V''(z,2) + \mu(2)V'(z,2) - (\lambda + \delta)V(z,2) = -\lambda(z - b(1) + V(b(1),1)),$$

with boundary conditions

$$V(0,1) = V(0,2) = 0 \quad (43)$$

$$V'(b(1),1) = V'(b(2),2) = 1 \quad (44)$$

and regularity conditions

$$V(b(1+),2) = V(b(1+),1) \quad (45)$$

$$V'(b(1+),2) = V'(b(1+),1). \quad (46)$$

Let $\mathbf{F}(z) = (\mathbf{X}_1, \mathbf{X}_2)e^{\mathbf{J}z}(\mathbf{X}_1^{-1}, -\mathbf{X}_2^{-1})^\top$ be the matrix solution of the system of differential equation

$$\frac{1}{2}\Delta_\sigma^2 \mathbf{F}''(z) + \Delta_\mu \mathbf{F}'(z) + \begin{pmatrix} -(\lambda + \delta) & \lambda \\ \lambda & -(\lambda + \delta) \end{pmatrix} \mathbf{F}(z) = \mathbb{O}$$

for $z \in I_1$, with the null condition at the 0 barrier, i.e. $\mathbf{F}(0) = \mathbb{O}$ and defining by $z_1 \leq z_2 \leq z_3 \leq z_4$ the four solution of the algebraic equation

$$\begin{aligned} z^4 &+ z^3 \left[\frac{2\mu(1)}{\sigma^2(1)} + \frac{2\mu(2)}{\sigma^2(2)} \right] \\ &+ z^2 \left[\frac{2\mu(1)}{\sigma^2(1)} \frac{2\mu(2)}{\sigma^2(2)} + (\delta - \lambda) \left(\frac{2}{\sigma^2(1)} + \frac{2}{\sigma^2(2)} \right) \right] \\ &+ z \left[(\delta - \lambda) \frac{4(\mu(1) + \mu(2))}{\sigma^2(1)\sigma^2(2)} \right] = \frac{4\delta(2\lambda - \delta)}{\sigma^2(1)\sigma^2(2)} \end{aligned}$$

the matrix $\mathbf{J} = \text{diag}[z_1, z_2, z_3, z_4]$ and the matrices \mathbf{X}_1 and \mathbf{X}_2 are given by

$$\begin{aligned} \mathbf{X}_1 &= \begin{pmatrix} -\frac{2\lambda/\sigma^2(2)}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} & -\frac{2\lambda/\sigma^2(2)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} \\ -\frac{2(\lambda-\delta)/\sigma^2(1)-z_1(2\mu(1)/\sigma^2(1)+z_1)}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} & -\frac{2(\lambda-\delta)/\sigma^2(1)-z_2(2\mu(1)/\sigma^2(1)+z_2)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} \end{pmatrix} \\ \mathbf{X}_2 &= \begin{pmatrix} -\frac{2\lambda/\sigma^2(2)}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} & -\frac{2\lambda/\sigma^2(2)}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} \\ -\frac{2(\lambda-\delta)/\sigma^2(1)-z_3(2\mu(1)/\sigma^2(1)+z_3)}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} & -\frac{2(\lambda-\delta)/\sigma^2(1)-z_4(2\mu(1)/\sigma^2(1)+z_4)}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} \end{pmatrix}. \end{aligned}$$

The above expressions are a rearrangement of formulas obtained by using the software Mathematica®. We have that

$$\vec{V}(z) = \mathbf{F}(z) (k_1, k_2)^\top \quad 0 \leq z \leq b(1)$$

$$V(z, 2) = k_3 f(z) + g(z) - \frac{1}{\lambda + \delta} V(b(1), 1) \quad b(1) < z \leq b(2)$$

where

$$g(z) = \frac{\lambda}{\lambda + \delta} z + \frac{\lambda \mu(2)}{(\lambda + \delta)^2} - \frac{\lambda}{\lambda + \delta} b(1)$$

and $f(z)$ being any solution of the differential system

$$\frac{1}{2} \sigma^2(2) f''(z) + \mu(2) f'(z, 2) - (\lambda + \delta) f(z) = 0 ,$$

with $b(1) < z \leq b(2)$ and boundary condition $f'(b(2)) = \delta/(\lambda + \delta)$. We choose the special solution that has

$$f(b(2)) = -\frac{1}{\Theta} f'(b(2)) = -\frac{\delta}{\lambda + \delta} \frac{\sigma^2(2)}{\mu(2)}$$

that can be written as

$$f(z) = -\frac{\delta}{\lambda + \delta} \frac{\sigma^2(2)}{\mu(2)} \exp\left(\Theta(b(2) - z)\right) \cosh\left(\Delta(b(2) - z)\right) ,$$

with $\Theta = \frac{\mu(2)}{\sigma^2(2)}$ and $\Delta = \sqrt{\Theta^2 + \frac{2(\lambda + \delta)}{\sigma^2(2)}}$.

To solve for the constants, k_1 , k_2 and k_3 we need to solve the following system of equations

$$\begin{aligned} \vec{V}'(b(1)-) &= (1, V'(b(1)+, 2))^\top \\ V(b(1)-, 2) &= V(b(1)+, 2) \end{aligned}$$

that is equal to

$$\mathbf{F}'(b(1)) (k_1, k_2)^\top = (1, k_3 f'(b(1)) + g'(b(1)))^\top \quad (47)$$

$$e_2^\top \mathbf{F}(b(1)) (k_1, k_2)^\top = k_3 f(b(1)) + g(b(1)) - \frac{1}{\lambda + \delta} e_1^\top \mathbf{F}(b(1)) (k_1, k_2)^\top \quad (48)$$

Using (47) we get

$$(k_1, k_2)^\top = \mathbf{F}'(b(1))^{-1} (1, k_3 f'(b(1)) + \lambda/(\lambda + \delta))^\top$$

and using the (48) we get

$$k_3 = -\frac{\lambda \mu(2) - (1, \lambda + \delta) \mathbf{F}(b(1)) \mathbf{F}'(b(1))^{-1} (\lambda + \delta, \lambda)^\top}{(\lambda + \delta)^2 f(b(1)) - (\lambda + \delta) f'(b(1)) (1, \lambda + \delta) \mathbf{F}(b(1)) \mathbf{F}'(b(1))^{-1} e_2} .$$

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